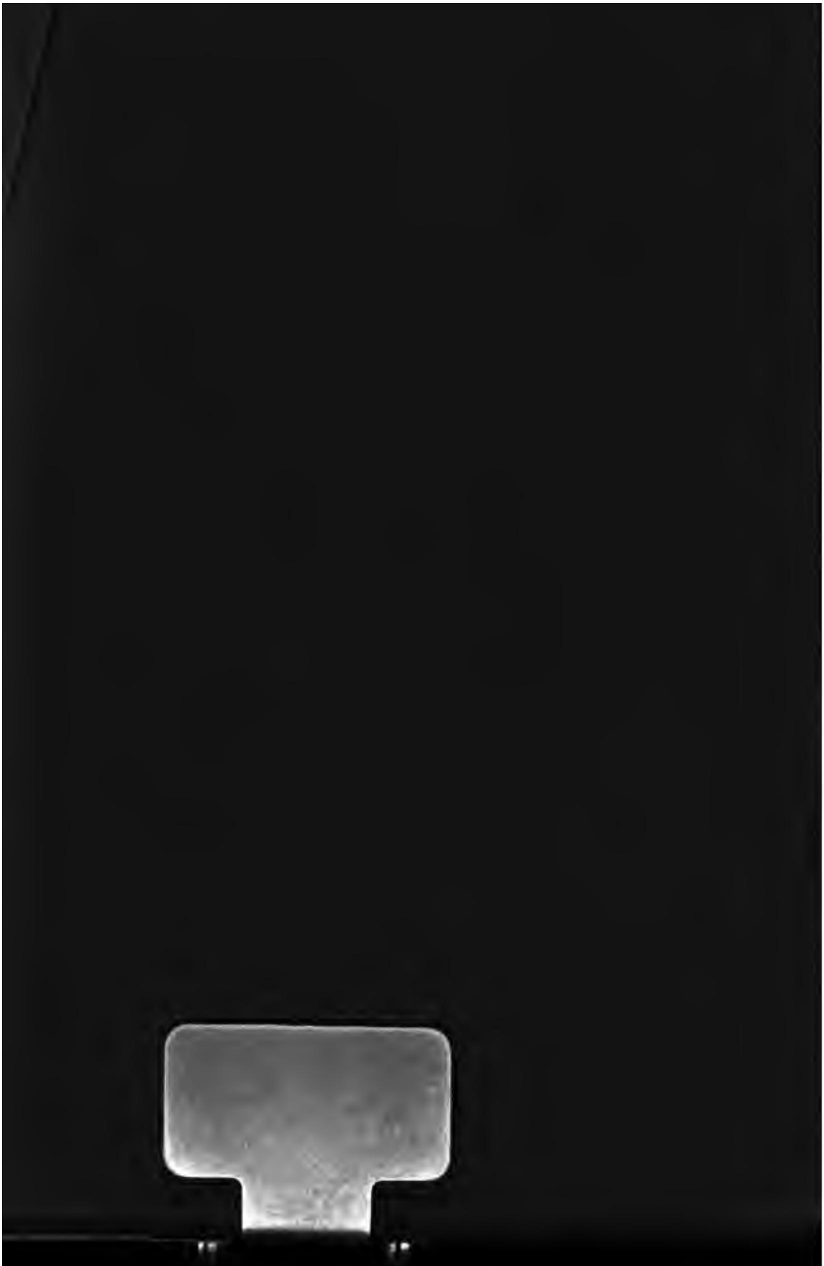
TEXT BOOKS OF SCIENCE

NOTES ON ALGEBRA AND TRIGONOMETRY

GRIFFIN



NOTES

ON THE ELEMENTS OF

ALGEBRA AND TRIGONOMETRY

WITH

SOLUTIONS of the more DIFFICULT QUESTIONS.

BY

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PREFACE.

THIS BOOK is intended to give some further assistance to students who are reading, without the aid of a teacher, the treatise on Algebra and Trigonometry in the Text Books of Science. As the living teacher who supplies help too liberally weakens the self-reliance of his pupils and hinders their intellectual growth, so this book may be a help or a hindrance according to the way in which it is used. It will be a hindrance to real progress in mathematical power when the student turns to it as soon as any difficulty meets him, to escape the exertion of patient thought on his own part; but it may be a help when it is used to suggest methods which the reader's own powers, after they have been honestly called into action, have failed to discover. Occasionally also this book may be useful in examples of the Text Book which the student has solved without assistance, if the methods which are here employed are more easy or concise than his own.

This work, therefore, is not a mere key, to give a detailed solution of every question which the Text

Book leaves as an exercise. The simpler examples in each group are not noticed, because if the reader cannot solve these he must be content to turn back and make his knowledge of preceding chapters more accurate.

It ought never to be forgotten that to see an example solved, and to follow the steps of the solution presented, is not always the gaining power to solve a similar question one's self, and that examples are worth little except as types of a class of examples which they represent. When a student has referred to this book to find the solution of a question which defeats him, after perusing the solution here given he should close the book and produce by himself on paper the solution which he has been examining. He may then do well to pause and endeavour to invent for himself other questions to which the same method of solution may be applied. He will thus have gained, through the example solved for him in this book, some new elements of power. Mathematical proficiency results, not from inspecting the solutions of a number of examples, but from the hours of patient thought spent in drawing from them the power of solving other similar examples which may arise.

The chapters and articles to which reference is made are those of the treatise on Algebra and Trigonometry in the Text Books of Science.

OSPRINGE VICARAGE: *November* 5, 1872.

NOTES

ON THE

Elements of Algebra and Trigonometry.

ALGEBRA.

CHAPTER I.

Art. **18.** Ex. 1. $a-b=20-10=10$;

$$-a+b=-20+10=-10.$$

The first statement means that the traveller has journeyed 10 miles to the east; the second statement that he has journeyed 10 miles away from the east, or towards the west.

Ex. 2. Expense being exhibited by a positive quantity, and leaving the man with this sum less, a negative quantity leaves him with so much more, and represents gain or income. When a day's expenses are $b-a=-$£10, the meaning is that on that day the man received £10, or ended the day richer by £10.

Ex. 3. Boys joining a school being represented by a positive number, boys leaving it are represented by a negative number. Thus $-a=-20$ new boys, signifies that 20 boys leave. Hence,

$$-a+b=-20+10=-10 \text{ new boys}$$

make the school smaller.

Ex. 4. $-b$ would mean 10 degrees below the freezing point.

Ex. 5. If one gains a games, the other must lose the same number, and loss is exhibited in contrast to gain by the negative sign. To win $b-a=-10$ games is to lose 10 games.

Ex. 6. If work of any kind is measured by a positive quantity, the negative sign would not denote inaction, but work of a reversing or antagonistic kind. If a workman builds 10 feet of a wall, his building -10 feet would express his unbuilding, or pulling down 10 feet, because the result of both acts in succession gives $10-10$ feet, or no wall resulting. Thus, in this example, if a means 20 days' work for a particular object, $-b=-10$ would mean 10 days' work for the contrary object, to destroy so far the results of the 20 days.

45. Ex. 12.
$$\begin{aligned} &x^2-y^2-(x-y)^2+a\{x-(y-z)\} \\ &=4^2-3^2-1^2+2\{4-2\} \\ &=16-9-1+2\times 2=10. \end{aligned}$$

Ex. 13. $\sqrt[3]{\dfrac{5(a^3+b)}{4(a^2-b)}}$. The operations here indicated are thus described:

1. Cube the quantity a.
   Add to this cube b.
   Multiply this sum by 5 and make the result the numerator of a fraction.
2. Square the quantity a.
   Subtract from this square b.
   Multiply this remainder by 4 and make the result the denominator of a fraction.
3. Of the fraction thus formed extract the cube root.

Ex. 14. $d-(c-b+a)=d-c+b-a$ $(42)=4-3+2-1=2$,
$d+c-(a+b)=d+c-a-b=4+3-1-2=4$.
$\therefore \{d-(c-b+a)\}\,\{d+c-(a+b)\}=2\times 4=8$.

Also

$$\begin{aligned} d^2-(c^2+b^2)+a^2+2(bc-ad) &= d^2-c^2-b^2+a^2+2bc-2ad \\ &= 16-9-4+1+12-8=8. \end{aligned}$$

CHAPTER II.

71. Ex. 17. Under the recommendation of (63) that the power should be gained as early as possible of effecting multiplications without an arrangement in vertical rows, Ex. 17 may thus be treated:

$$\begin{aligned}(x+\cdot3)(x+\cdot16) &= x^2+\cdot3x+\cdot16x+\cdot048\\ &= x^2+\cdot46x+\cdot048.\\ \therefore (x+\cdot3)(x+\cdot16)(x+1\cdot25) &= (x^2+\cdot46x+\cdot048)(x+1\cdot25)\\ &= x^3+\cdot46x^2+\cdot648x\\ &\quad +1\cdot25x^2+\cdot575x+\cdot06\\ &= x^3+1\cdot71x^2+\cdot623x+\cdot06.\end{aligned}$$

Ex. 23. Though a learner may be well contented for a time to bring out correct results by methods more obvious to him, he may gain instruction by observing how materially this example is made easier by a proper combination of the factors.

$$\begin{aligned}&(x-3a)(x-a)(x+a)(x+3a)\\ &\quad= (x-3a)(x+3a)(x-a)(x+a)\\ &\quad= (x^2-9a^2)(x^2-a^2) \quad (64)\\ &\quad= x^4-9a^2x^2+a^2x^2+9a^4\\ &\quad= x^4-10a^2x^2+9a^4.\end{aligned}$$

This suggestion may be remembered with advantage in Ex. 25.

Ex. 32. An example of this kind may be conveniently treated by multiplying out each of the equated expressions, and showing the identity of the results.

Thus $\quad (x+y+z)(yz+xz+xy)-xyz$
$$= x^2y+x^2z+y^2x+y^2z+z^2x+z^2y+2xyz$$

and $(y+z)(z+x)(x+y)$ when multiplied out produce the same expression.

82. Ex. 33. The coefficient of x in the dividend is first to be exhibited explicitly.

Since $\quad (a^2+b^2)(a-b)=a^3-a^2b+ab^2-b^3,$

the operation will now stand

$$
\begin{array}{llr}
x^2-(a-b)x+b^2) & & \overset{(1)}{(x^2}+\overset{(2)}{(a-b)x}-\overset{(3)}{a^2} \\
x^4-2a(a-b)x^2+(a^3-a^2b+ab^2-b^3)x-a^2b^2 & & \\
\underline{x^4-(a-b)x^3+b^2x^2} & & (1) \\
(a-b)x^3-(2a^2-2ab+b^2)x^2+(a^3-a^2b+ab^2-b^3)x-a^2b^2 & & \\
\underline{(a-b)x^3-(a^2-2ab+b^2)x^2+(ab^2-b^3)x} & & (2) \\
\qquad -a^2x^2 \qquad\qquad +(a^3-a^2b)x-a^2b^2 & & \\
\underline{\qquad -a^2x^2 \qquad\qquad +(a^3-a^2b)x-a^2b^2} & & (3) \\
\qquad\qquad \underline{\cdot \quad \cdot \quad \cdot} & &
\end{array}
$$

The references (1) (2) (3) connect the separate terms of the quotient with the subtrahends respectively produced by them.

88. Ex. 10.

$$
\begin{aligned}
&(ac-bd)^2+(ad+bc)^2 \\
&=a^2c^2-2abcd+b^2d^2 \\
&+a^2d^2+2abcd+b^2c^2 \\
&=a^2c^2+a^2d^2+b^2d^2+b^2c^2 \\
&=a^2(c^2+d^2)+b^2(d^2+c^2) \\
&=(a^2+b^2)(c^2+d^2).
\end{aligned}
$$

Ex. 11. The expression presented is

$$(a-b)^2x-a^2+2abx-b^2x^2.$$

The coefficient (20) of x is

$$
\begin{aligned}
&(a-b)^2+2ab \\
&=a^2-2ab+b^2+2ab \\
&=a^2+b^2.
\end{aligned}
$$

Ex. 19. If in $x+y+z$ we regard $(x+y)$ as a single quantity, and so view $x+y+z$ as a binomial according to the suggestion in (87),

$$
\begin{aligned}
(x+y+z)^2 &= (x+y)^2+2(x+y)z+z^2\\
&= (x+y)^2+2xz+2yz+z^2.
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ (x+y+z)^2&+x^2+y^2+z^2\\
&=(x+y)^2+x^2+2xz+z^2+y^2+2yz+z^2\\
&=(x+y)^2+(x+z)^2+(y+z)^2 \quad (84.\ \text{i}).
\end{aligned}
$$

Ex. 20. By the method of (87),

$$
\begin{aligned}
(x+y+z)^3 = x^3+y^3+z^3&+3x^2y+3x^2z\\
&+3y^2x+3y^2z\\
&+3z^2x+3z^2y\\
&+6xyz.
\end{aligned}
$$

Now by performing the multiplication indicated it will be found that

$$(y+z)(z+x)(x+y)=x^2y+x^2z+y^2x+y^2z+z^2x+z^2y+2xyz.$$
$$\therefore\ (x+y+z)^3=x^3+y^3+z^3+3(y+z)(z+x)(x+y).$$

Ex. 22. Treating $x+y$ as a single quantity so as to view $x+y+z$ as a binomial, according to the method of (87), we have

$$
\begin{aligned}
(x+y+z)^2 &= (x+y)^2+2(x+y)z+z^2\\
&= x^2+y^2+z^2+2yz+2xz+2xy \quad (84.\ \text{i}).\\
\therefore\ 4a^2 &= x^2+y^2+z^2+2yz+2xz+2xy.
\end{aligned}
$$

Now from this expression the square of each of the succeeding polynomials (34) may immediately be written down, when it will be observed that they differ from $x+y+z$ in the sign of one of the quantities x, y, z.

In the value of $2b$, x has received a negative sign.

$$
\begin{aligned}
\therefore\ 4b^2 &= x^2+y^2+z^2+2yz-2xz-2xy. \quad (31)\\
\text{So } 4c^2 &= x^2+y^2+z^2-2yz+2xz-2xy\\
\text{and } 4d^2 &= x^2+y^2+z^2-2yz-2xz+2xy.
\end{aligned}
$$

If now the four quantities, $4a^2$, $4b^2$, $4c^2$, $4d^2$, be added

together, their sum must be equal to the sum of the several expressions to which they are separately equal.

$$4a^2+4b^2+4c^2+4d^2=4x^2+4y^2+4z^2,$$

and as these quantities are equal, so must the quantities be equal which result from dividing each of them by 4.

$$\therefore\ a^2+b^2+c^2+d^2=x^2+y^2+z^2.$$

It will be observed in this example, and it is worthy of remembrance, that when the expression for $(x+y+z)^2$ or $(x+y+z)^3$ is obtained, we can immediately write down from it the expressions for $(x+y+z)^2$, $(x-y-z)^3$, and all others formed by changing the signs only of some of the symbols. Thus, when it shall have been found that

$$(x+y+z)^3=x^3+y^3+z^3+3x^2y+3x^2z+3y^2x+3y^2z \\ +3z^2x+3z^2y+6xyz$$

it will be at once observed that

$$(x-y-z)^3=x^3-y^3-z^3-3x^2y-3x^2z+3y^2x-3y^2z \\ +3z^2x-3x^2v+6xyz$$

118. Ex. 8. $$\frac{x^3}{y}-\frac{y^3}{x}=\frac{x^4-y^4}{xy},$$

and $$\frac{x}{y}-\frac{y}{x}=\frac{x^2-y^2}{xy}.$$

Hence the division intended gives

$$\frac{x^4-y^4}{xy}\ \ \frac{xy}{x^2-y^2}=\frac{x^4-y^4}{x^2-y^2}=x^2+y^2\quad(64).$$

Examples 9, 10, and 12 may be similarly treated.

Ex. 11. If the expression to be divided is brought to one denominator it becomes

$$\frac{ax^3-a^2x^2-abx^2-b^2x^2+a^2bx+a^2b^2}{a^2x^2}$$

and the result arises from dividing the numerator by $x-a$.

122. Ex. 3. Regard $\frac{a}{b}+\frac{b}{a}$ as a single quantity according to the recommendation of (87).

Thus $\left(\frac{a}{b}+\frac{b}{a}-\frac{1}{2}\right) = \left(\frac{a}{b}+\frac{b}{a}\right)^2 - \left(\frac{a}{b}+\frac{b}{a}\right)+\frac{1}{4}.$

Now $$\left(\frac{a}{b}+\frac{b}{a}\right)^2 = \frac{a^2}{b^2}+2\frac{ab}{ba}+\frac{b^2}{a^2}$$
$$=\frac{a^2}{b^2}+2+\frac{b^2}{a^2},$$

∴ the result required is

$$\frac{a^2}{b^2}+2+\frac{b^2}{a^2}-\frac{a}{b}-\frac{b}{a}+\frac{1}{4}$$
$$=\frac{a^2}{b^2}+\frac{b^2}{a^2}-\frac{a}{b}-\frac{b}{a}+\frac{9}{4}.$$

CHAPTER III.

148. Ex. 10. This is the only equation in this group which may present any difficulty—

$$m^2-n-mx=n^2-m-nx.$$

By transposition

$$nx-mx=n^2-m^2+n-m,$$
$$(n-m)x=n^2-m^2+n-m.$$

Dividing by $n-m$ we have

$$x=n+m+1.$$

153. Ex. 29. When the whole equation has been multiplied by 12 to remove the fractions, it gives

$$\begin{aligned}3x-3m^2+2x-2n^2 &= 7mn,\\ 5x &= 3m^2+7mn+2n^2\\ &= 3m^2+mn+6mn+2n^2\\ &= (3m+n)m+2(3m+n)n\\ &= (3m+n)(m+2n).\end{aligned}$$

$$\therefore\ x=\frac{(3m+n)(m+2n)}{5}.$$

164. Ex. 16. Since $\dfrac{x+4a+b}{x+a+b}=1+\dfrac{3a}{x+a+b}$,

and $\dfrac{4x+a+2b}{x+a-b}=4+\dfrac{6b-3a}{x+a-b}$,

$$\therefore 5+\frac{3a}{x+a+b}+\frac{6b-3a}{x+a-b}=5,$$

whence $\dfrac{3a}{x+a+b}=\dfrac{3(a-2b)}{x+a-b}$,

$$\frac{x+a+b}{a}=\frac{x+a-b}{a-2b},$$

$$\begin{aligned}(x+a)(a-2b)+ab-2b^2 &= (x+a)a-ab,\\ 2b(x+a) &= 2(ab-b^2),\\ x+a &= a-b,\\ x &= -b.\end{aligned}$$

This solution is less laborious than the more obvious one of multiplying the equation by the denominators of the fractions. The method may be applied in Examples 19 and 22.

Ex. 24. If x has the value intended,

$$(2x+4)(3x+4)-(3x-2)(2x-8)=96,$$

and by solution of this equation $x=2$.

Ex. 25. In this case n is the quantity whose value is required, just as x has been heretofore.

$$\frac{2a+n}{3n+69a} = \frac{1}{33},$$

$$\frac{2a+n}{n+23a} = \frac{1}{11},$$

$$22a+11n = n+23a,$$

$$10n = a,$$

$$n = \frac{a}{10}, \text{ and therefore } = \frac{1}{30} \text{ when } a = \frac{1}{3}.$$

Ex. 26. $$\frac{(x-a)(x-b)}{x-a-b} = \frac{(x-c)(x-d)}{x-c-d},$$

or $$\frac{x^2-ax-bx+ab}{x-a-b} = \frac{x^2-cx-dx+cd}{x-c-d},$$

$$\therefore x + \frac{ab}{x-a-b} = x + \frac{cd}{x-c-d},$$

$$\therefore \frac{ab}{x-a-b} = \frac{cd}{x-c-d},$$

$$\frac{x-a-b}{ab} = \frac{x-c-d}{cd},$$

whence $$x = \frac{ab(c+d)-cd(a+b)}{ab-cd}.$$

170. Ex. 7. In this example the members of the equation may have to be squared more than once to obtain x.

$$1+2\sqrt{x} = \sqrt{4x+\sqrt{16x+2}}.$$

If each member be squared,

$$1+4\sqrt{x}+4x = 4x+\sqrt{16x+2},$$

or $$1+4\sqrt{x} = \sqrt{16x+2}.$$

Again, if each member of the equation thus produced be squared,

$$1+8\sqrt{x}+16x = 16x+2,$$
$$8\sqrt{x} = 1,$$
$$\sqrt{x} = \frac{1}{8}, \quad x = \frac{1}{64}.$$

Ex. 12 will require similar treatment.

Ex. 10. Since $(\sqrt{5x}+3)(\sqrt{5x}-3) = 5x-9$,

$$\frac{5x-9}{\sqrt{5x}+3} = \sqrt{5x}-3.$$

$$\therefore \sqrt{5x}-3 = 1+\frac{\sqrt{5x}-3}{2},$$

whence $\sqrt{5x} = 5$, $x = 5$.

Ex. 11. This is a generalized form of equation to which the equation solved in (167) is similar.

$$\sqrt{x+p} = \sqrt{p+q} - \sqrt{x+q},$$
$$x+p = p+q-2\sqrt{p+q}\sqrt{x+q} + x+q,$$
$$\sqrt{p+q}\sqrt{x+q} = q,$$
$$x+q = \frac{q^2}{p+q},$$
$$x = \frac{q^2}{p+q} - q = -\frac{pq}{p+q}.$$

CHAPTER IV.

188. Ex. 18. The expression 'half as much again' is thus algebraically exhibited. If a be any quantity, its half is $\frac{a}{2}$. 'Half as much again' means the addition of the half to the

original quantity, and gives us therefore $a+\frac{a}{2}$ or $\frac{3a}{2}$. Thus half as much again of a quantity is expressed by multiplying the quantity by $\frac{3}{2}$. Similarly 'a third as much again' would be exhibited by multiplying the quantity in question by $\frac{4}{3}$.

In this instance, if B has x£ and A consequently has $3x$£, after the transfer A has $3x-150$ £ and B has $x+150$ £. The terms of the question then give

$$3x-150=\frac{3}{2}(x+150),$$
$$\text{whence } x=250,$$
$$3x=750.$$

Ex. 19. Let x be the unknown larger number, the smaller being known to be 4.

$$\text{Their sum is } x+4,$$
$$\text{,, difference is } x-4.$$
$$\therefore x+4=2(x-4),$$
$$x=12.$$

Ex. 22. If $2x$ apples are bought, since each costs $\frac{2}{5}$ of a penny, they cost $\frac{4x}{5}$ pence.

Now in the first sale x are sold at 2 a penny, or at $\frac{1}{2}d.$ each, therefore for $\frac{x}{2}$ pence.

And in the second sale x are sold at 3 a penny, or at $\frac{1}{3}d.$ each, therefore for $\frac{x}{3}$ pence.

The sums received by the sales exceed the cost price by one penny.

$$\therefore \frac{x}{2}+\frac{x}{3}=1+\frac{4x}{5},$$
$$\text{whence } x=30,$$

and $2x=60$, the number of apples bought.

Ex. 26. Let $x+54$ be the number required,

$\therefore$ $x-54$ the number 108 less.

$$\therefore \frac{x+54}{8} - \frac{x+54}{12} = 1 + \frac{x-54}{9} - \frac{x-54}{15},$$

whence $x = 1314$,

and the number $x+54$ is 1368.

Ex. 27. Let there be $27+x$ loads in the larger stack,
$27-x$ loads in the smaller.

The remainder of the cut stack has $27-x-12$ or $15-x$ loads.

$$\therefore 27+x = 2(15-x),$$
whence $x = 1$,

and the stacks have $27+x=28$, and $27-x=26$ loads, respectively.

Ex. 28. To avoid fractions let the income be $35x$ £ yearly.

A's annual debt is $\frac{1}{7}35x$ £ or $5x$ £,
and his debt in 10 years is $50x$ £.
B's annual saving is $35x$ £$-\frac{4}{5}35x$ £ or $7x$ £,
and his saving in 10 years is $70x$ £.

This saving of B suffices to pay A's debts and to leave 160 £.

$$\therefore 70x = 160+50x,$$
$$x = 8,$$

and the income of A or B is $35x$ or 280 £.

Ex. 29. Let $5x+5$ £ be the original sum.

1. A receives $5+x$ £.
2. After B has also received 10 £ there remains

$$5x+5-(5+x)-10 = 4x-10 \text{ £},$$

$\therefore$ B receives in all $10 + \frac{4x-10}{5}$ £.

These sums which A and B receive, together with 15 £ which C receives, make up the whole sum distributed.

$$\therefore\ 5+x+10+\frac{4x-10}{5}+15=5x+5,$$

whence $x=\frac{115}{16}$,

$$5x=\frac{575}{16}\text{£}=\text{£}35\ 18s.\ 9d.,$$

and the sum required $5x+5=\text{£}40\ 18s.\ 9d.$

Ex. 30. The property of consecutive numbers being that one exceeds the other by unity, let the numbers required be denoted by x and $x+1$;

$\therefore$ their squares being x^2 and $(x+1)^2$,

$$(x+1)^2-x^2=15,$$
$$x=7,$$

and the numbers are 7 and 8.

Ex. 32. An equation will be obtained by finding, from the sums at which the portions are sold, the price of a yard of either, a common element.

The relation of the lengths of the two portions is expressed by supposing these lengths to be $5x$ and $6x$ yards.

After they are reduced the remainders are

$$\left.\begin{matrix}5x-10\\6x-10\end{matrix}\right\}\text{yards.}$$

Since $5x-10$ yards are sold for 10 half-sovereigns, the price of a yard is $\frac{10}{5x-10}$ half-sovereigns.

Since $6x-10$ yards are sold for 13 half-sovereigns, the price of a yard is $\frac{13}{6x-10}$ half-sovereigns.

$$\therefore\ \frac{10}{5x-10}=\frac{13}{6x-10},$$

whence $x=6$.

The lengths of the pieces are $5 \times 6 = 30$ yards, and $6 \times 6 = 36$ yards, and the price of a yard $\dfrac{10}{5x-10} = \frac{1}{2}$ half-sovereign.

$$= 5s.$$

Ex. 36. Sixpence in the pound is $\frac{1}{40}$th of the sum taxed. Let the original income be $40x$ £.

The remainder after the tax paid is $39x$ £.

$$\begin{aligned} \therefore 39 - \tfrac{1}{13} \cdot 39x &= 540, \\ \text{or } 36x &= 540, \\ 4x &= 60, \\ 40x &= 600, \end{aligned}$$

and £600 is the original income.

Ex. 37. The number of half-crowns were to be even, or there would be an odd sixpence, and the sum to be paid is 74 shillings. Let there be $2x$ half-crowns used, and consequently $41-2x$ shillings.

The $2x$ half-crowns being worth $5x$ shillings,

$$\begin{aligned} 5x + 41 - 2x &= 74, \\ x &= 11, \end{aligned}$$

$2x = 22$ the number of half-crowns,

$41 - 2x = 19$ „ shillings.

196. Ex. 47. When the publication is made with the least number of copies which can be issued without loss, the loss of a farthing in each copy must be balanced by the receipts for advertisements.

Suppose $10{,}000(x+1)$ copies issued.

The loss is $10{,}000(x+1)$ farthings.

From advertisements there are $3{,}000\,x$ pence, or $12{,}000\,x$ farthings received.

$$\begin{aligned} \therefore 12{,}000\,x &= 10{,}000(x+1), \\ x &= 5. \end{aligned}$$

$\therefore$ 60,000 copies are issued.

It will be observed how the assumption here made to represent the number of copies issued is adopted in order to avoid any fractions in the equation.

201. Ex. 54. Two processes of alteration are here to be considered.

Let the larger vessel originally contain $2x$ gallons.

$\therefore$ the smaller „ contains $2x-21$ gallons.

I. The larger loses $x+4$ gallons, and gains from the smaller

$$\frac{2x-21}{2}-1\tfrac{1}{2}=12 \text{ gallons},$$

$\therefore$ the larger now contains $2x-16$ gallons,
the smaller „ $x-9$ gallons.

II. The second alteration makes

the larger to contain $2x-16+4\frac{1}{2}=2x-11\frac{1}{2}$ gallons,
the smaller „ $x-9-4 = x-13$ gallons.

By the condition of the problem

$$2x-11\tfrac{1}{2}=2\tfrac{1}{2}(x-13),$$

whence
$$x=42,$$
$$2x=84,$$

the number of gallons which the larger cask contains, the smaller accordingly containing 63.

209. Ex. 63. Since the intervals of time are quarters of an hour, it is convenient to take a quarter of an hour as the unit of time, and to suppose that the train at its average rate travels x miles every quarter of an hour.

Hence, the train from A to B, 70 miles, is $\frac{70}{x}$ quarters of an hour.

„ „ B to C, 56 miles, is $\frac{56}{x}$ quarters of an hour.

Since there is a stoppage of 1 quarter of an hour at B, the whole journey takes $\frac{70}{x} + 1 + \frac{56}{x}$ quarters of an hour.

$\therefore$ by the condition of the problem,

$$\frac{70}{x} + 1 + \frac{56}{x} = 2 . \frac{70}{x} - 1,$$

$$\frac{70-56}{x} = 2, \quad x = 7,$$

or the average rate is 7 miles every quarter of an hour, and 28 miles an hour.

Ex. 64. The alteration stated affects the latter half of the journey only, which is usually performed in 2 hours, but at the increased speed is performed in 1 hour and 28 minutes, or $\frac{22}{15}$ hours.

Let the ordinary speed be $x-4$ miles an hour,
 „ the increased „ $x+4$ „

In one view, then, the latter half of the journey must be $2(x-4)$ miles, and in the other view it must be $\frac{22}{15}(x+4)$ miles.

$$\therefore \frac{22}{15}(x+4) = 2(x-4), x = 26.$$

$\therefore$ the ordinary speed is $x-4$, or 22 miles an hour, and the distance from London to Dover, being performed at this rate in 4 hours, is 88 miles.

Ex. 65. Problems of this kind are solved by remembering that, while each hand of a clock moves uniformly, the minute hand moves twelve times as fast as the hour hand.

The reader who has found difficulty in this example may be assisted by previously considering the following simpler instance in which the same principles come into use, but one clock only is under consideration.

What is the time between 2 and 3 o'clock when the minute hand of a clock is 12 minute spaces in advance of the hour hand?

Let the time required be $12x$ minutes after 2 o'clock.

If we regard the twelfth hour mark of the clock as our starting-point, when both the hands were in coincidence, the minute hand has advanced $12x$ minute spaces from this point since 2 o'clock. In this time the hour hand has advanced x minute spaces from the second hour mark on the face, or is $10+x$ minute spaces from the twelfth hour mark. Since the minute hand, as we are informed, is 12 minute spaces before the hour hand,

$$12x = 10+x+12$$
$$x = 2,$$
$$12x = 24,$$

and the time is 24 minutes after 2 o'clock.

To return to Ex. 65, though each clock goes incorrectly for the purpose of marking time, yet in each there is uniform rate of motion of the hands.

The minute hand of the clock A passes over 75 minute divisions in an hour, or $\frac{75}{60} = \frac{5}{4}$ in a minute.

The minute hand of B passes over 63 minute divisions in an hour, or $\frac{63}{60} = \frac{21}{20}$ in a minute.

Let the time required be x minutes past 12.

In the time since noon the hand of A has travelled over $\frac{5x}{4}$ minute spaces, and being at noon right, appears $\frac{5x}{4}-x$ minutes too fast; while the hand of B has travelled over $\frac{21x}{20}$ minutes spaces, and having been 21 minutes too fast at noon, is now $21+\frac{21}{20}x-x$ minutes too fast. By the condition of the question,

$$21+\frac{21}{20}x-x = 2\tfrac{3}{10}\left(\frac{5x}{4}-x\right),$$

whence $x=40$,

or the time required is 40 minutes past 12.

At this time A is 10 minutes too fast, B is 23 minutes too fast, and $\frac{23}{10} = 2\frac{3}{10}$.

Ex. 66. A requires for his work 3 hours more than B.

Let A require $2x$ hours,
and B ,, $2x-3$ hours.

Therefore they do in an hour $\frac{1}{2x}$ and $\frac{1}{2x-3}$ of the work respectively. When A has done half his work, he has been x hours working, and B has been $x-2$ hours. B, therefore, has finished $\frac{x-2}{2x-3}$ of his work, and leaves unfinished $\frac{x-1}{2x-3}$ of it.

B now takes the remaining half of A's work, and completes it in $x-\frac{3}{2}$ hours.

A has to do $\frac{x-1}{2x-3}$ of the work, and completes this in $2x\cdot\frac{x-1}{2x-3}$ hours.

Now B's time is less than A's by $121\frac{1}{2}$ minutes, or $\frac{121 \cdot 5}{60}$ hours.

$$\therefore\ 2x\frac{x-1}{2x-3} = x-\frac{3}{2}+\frac{121 \cdot 5}{60},$$

whence $2x = 63$,

so that A requires 63 hours and B requires 60 hours for the work.

CHAPTER V.

224. Ex. 1. $3\times(1)+4\times(2)$ gives x.
Ex. 2. $9\times(1)+5\times(2)$ gives y.
Ex. 3. $6\times(1)+(2)$ gives x.
Ex. 4. $2\times(1)+3\times(2)$ gives x.

After these examples it is hoped that the method of solution to be adopted in the rest will be readily observed.

When several fractions appear in either or both of the equations, they must be simplified in the manner used with equations of one variable. For instance,

Ex. 16. $$\frac{7x+3y}{11}-\frac{4x-5y}{5}=\frac{15-y}{4}.$$

$$140x+60y-176x+220y=825-55y,$$
$$-36x+335y=825, \quad (1)$$
and $$3x-5y=0. \quad (2)$$
$$\therefore (1)+12\times(3) \text{ gives } 275y=825,$$
$$y=3;$$
and then $x=5$.

229. Ex. 2. $(2)-2\times(1)$ gives $-10y+11z=-26$. (4)
$(3)-3\times(1)$ gives $-5y+8z=-3$,
whence $-10y+16z=-6$. (5)
$(5)-(4)$ gives $5z=20$, $z=4$;
whence $y=7$, $x=3$.

Ex. 3. Though six equations may be formed from this statement, there are but three independent equations, because when three are formed the remaining three may be produced from them.

We have $9-4x+3=5x+6y+18$,
whence $3x+y=9$, (1)
and $27-12x+9y=4x+10y-8$,
whence $16x+y=35$. (2)
$(2)-(1)$ gives $13x=26$, $x=2$,
and then $y=3$, $z=5$.

Ex. 4. Convert the equations into the forms

$$\frac{1}{y}+\frac{1}{z}=\frac{7}{36},$$
$$\frac{1}{z}+\frac{1}{x}=\frac{1}{4},$$
$$\frac{1}{x}+\frac{1}{y}=\frac{5}{18}.$$

The equations are now substantially of the same form with those of Ex. 1, if reciprocals are viewed as taking the placcs of the quantities to be found.

$$\text{Hence}\quad \frac{1}{x}=\frac{1}{6},\quad \frac{1}{y}=\frac{1}{9},\quad \frac{1}{z}=\frac{1}{12},$$

$$\text{and}\quad x=6,\quad y=9,\quad z=12.$$

Ex. 5 is to be treated on the same method.

CHAPTER VI.

238. The following are the only examples of this group in which it is supposed that assistance may be offered with advantage to the reader.

Ex. 11. If x be the right-hand digit and y the other digit, since the digit y in this position means $10y$ units, the number is $10y+x$ units. When the digits are reversed, or y becomes the right-hand digit, and x, meaning $10x$ units, becomes the left-hand digit, the number is $10x+y$ units.

$$\text{Hence}\quad 10y+x+10x+y=121,$$
$$10y+x-(10x+y)=9,$$
$$\text{or } x+y=11,\quad (1)$$
$$y-x=1,\quad (2)$$
$$\text{whence } x=5, y=6,$$
and the number is 65.

Ex. 12. The value of a barrel of beer is first to be ascertained. Let it be x£, while the value of a barrel of porter is y£.

$$\text{Hence}\quad 6x+10y=51,\quad (1)$$
$$3x+7y=32.\quad (2)$$

$$7\times(1)-10\times(2) \text{ gives } 12x=36,\ x=3.$$

If then a barrel of beer is worth 3£, for £30 ten barrels can be bought.

Ex. 14. The values of the symbols a and b have first to be determined, and for this purpose we know that

$$2a+b=30, \quad (1)$$
$$5a+b=90. \quad (2)$$
$$(2)-(1) \text{ gives } 3a=60,\ a=20,$$
$$\text{and then } b=30-2a=-10.$$
$$\therefore\ ax+b=23x-10.$$

The value of this expression when x is 3·5 is $70-10=60$,

and it is zero when $x=\frac{10}{20}$ or ·5.

Ex 17. The value of the money in the bag shows that there must be an even number of crowns. Let that number be $2x$, and their value accordingly x half-sovereigns.

Let there be y half-sovereigns in the bag.

$$\therefore\ y+x=1{,}257, \quad (1)$$

the value of the money in half-sovereigns.

Now 623 half-sovereigns weigh as much as 88 crowns,

1	"	"	"	$\frac{88}{623}$
y	"	"	"	$\frac{88y}{623}$

and $\frac{88y}{623}$ crowns are worth $\frac{44y}{623}$ half-sovereigns,

$$\therefore\ x+\frac{44y}{623}=99. \quad (2)$$

$$(1)-(2) \text{ gives } \frac{579}{623}y=1158,$$
$$y=1246,$$
$$x=1257-1246=11,$$

and $2x$ the number of crowns is 22.

Ex. 18. Let each gold coin be worth x shillings,

" silver " y "

Each person takes 17 coins.

A takes 7 gold coins, and therefore 10 silver coins.

B then must take 10 gold and 7 silver coins.

A's share is worth 120 shillings.

B's " 135 "

$$\text{Hence } 7x+10y=120, \quad (1)$$
$$10x+7y=135, \quad (2)$$
$$\text{whence } x=10,\ y=5,$$

and the gold coins being worth 10 shillings each are half-sovereigns, the silver coins are crowns.

Ex. 19. Let the distance between A and B be $2x+3$ miles,

∴ " C and D is $2x-3$ "

and let " B and C be $2y$ "

∴ the whole distance AD is $4x+2y$ miles,

" AC is $2x+2y+3$ "

" BD is $2x+2y-3$ "

$$\therefore\ 2x+2y+3=3x+3y-4\tfrac{1}{2}-\tfrac{1}{16},$$
$$\text{or } x+y=\tfrac{121}{16}. \quad (1)$$

Since the distance between A and D is $4x+2y$ miles, the middle point between A and D is $2x+y$ miles from A,

$$\therefore\ 2x+y=2x+3+\tfrac{1}{2}, \quad (2)$$
$$\text{or } y=3\tfrac{1}{2},$$
$$\text{and then } x=\tfrac{65}{16}.$$

∴ the distance from A to B $=2x+3=11\frac{1}{8}$ miles,

" B to C $=2y=7$ "

" C to D $=2x-3=5\frac{1}{8}$ "

Ex. 20. Let x be the number of octavos and y the number of duodecimos to be determined. If the size of the table be unity,

since an octavo occupies $\frac{1}{x}$ of the table,

and a duodecimo „ $\frac{1}{y}$ „

$$\therefore \frac{15}{x} + \frac{12}{y} = 1, \quad (1)$$

$$\frac{9}{x} + \frac{8}{y} = \frac{5}{8}, \quad (2)$$

$$\text{whence } \frac{1}{x} = \frac{1}{24}, \; x = 24,$$

$$\frac{1}{x} = \frac{1}{32}, \; y = 32.$$

Ex. 21. In a gallon of the mixture let x be the fractional part of the gallon which is wine, and y the fractional part of the gallon which is brandy.

Since these fractions together form a gallon,

$$x + y = 1 \quad (1)$$

Now 28*s*. with the addition of 15 per cent. is 32·2 shillings,
„ 42*s*. „ 20 „ 50·4 „

∴ at these increased prices the component parts of a gallon of mixture are worth 32·2 x and 50·4 y shillings respectively.

$$\therefore 32{\cdot}2\,x + 50{\cdot}4\,y = 38 \quad (2)$$

$$\text{Hence} \quad x = \tfrac{62}{91}, \; y = \tfrac{29}{91}.$$

If then a gallon be divided into 91 equal parts, in the mixture described 29 of these parts are brandy and 62 of these parts are wine.

Ex. 22. In this question the mention of the interval of six months, introduced, it may be, to give an air of reality to the description, has no bearing on the solution. Two separate sellings out are supposed, it matters not when, and the circumstances of each supply an equation. The words 'by

sale' remove all consideration of interest or dividend arising from the shares while they are held.

It will be convenient to avoid fractions by supposing that

$10x$ £ are laid out in the shares of the first company,
$30y$ £ „ „ second company.

I. In the first supposed sale—

The loss by shares of the first company $= 5x - 35$ £,
„ „ second „ $= 10y + 11\frac{2}{3}$ £,
$\therefore$ the whole loss $= 5x + 10y - \frac{70}{3}$.

We are otherwise informed that this loss is 10 £ more than $\frac{3}{8}$ of the whole original outlay, or $10 + \frac{3}{8}(10x + 30y)$.

$$\therefore\ 5x + 10y - \tfrac{70}{3} = 10 + \tfrac{3}{8}(10x + 30y),$$
$$\text{whence } 3x - 3y = 80. \quad (1)$$

II. In the actual sale—

The loss by shares of the first company $= 2x - 10$ £,
the gain „ second „ $= 3y - 14$ £,
$\therefore$ the balance of loss $= 2x - 3y + 4$ £.

This loss is otherwise known to be $94 - \frac{1}{20}(10x + 30y)$ £,

$$\therefore\ 2x - 3y + 4 = 94 - \frac{x}{2}(x + 3y),$$
$$\text{whence } 5x - 3y = 180. \quad (2)$$

Now $(2) - (1)$ gives $2x = 100$,
and $10x = 500$,

and the outlay in the first company is £500.

Then $3y = 5x - 180 = 70$,
and $30y = 700$,

and the outlay in the second company is £700.

Ex. 23. Let P travel x miles an hour, and Q travel y miles an hour.

The village then is $x - 2$ miles from P's house, and he

goes to it in $\frac{x-2}{x}$ hours, and returns in the same time, and then travels 5 miles to the station in $\frac{5}{x}$ hours, so that when he arrives at the station it is $2\frac{x-2}{x}+\frac{5}{x}$ or $\frac{2x-1}{x}$ hours after 9 o'clock, and Q consequently has been travelling $\frac{2x+1}{x}-1\frac{1}{2}$ hours, allowing for his starting at half-past ten.

Now P's journey has been $2(x-2)+5$ or $2x+1$ miles.
∴ Q's „ $\frac{3}{2}(2x+1)-\frac{3}{2}=3x$ „

and Q has travelled this distance in $\frac{2x+1}{x}-\frac{3}{2}$ hours.

$$\therefore\ 3x = \left(\frac{2x+1}{x}-\frac{3}{2}\right)y$$
$$= \left(\frac{1}{2}+\frac{1}{x}\right)y. \quad (1)$$

Again, the relation given between the rates of travelling of P and Q is expressed by another equation,

$$y = 6x-8; \quad (2)$$
hence
$$x = 4,$$
$$y = 16.$$

CHAPTER VII.

261. All the earlier examples of this group are solved by application of the definite directions of (248), after the multiplications indicated in the example have been performed.

Ex. 16 requires treatment thus:

$$\frac{x}{a}+\frac{a}{x}=\frac{x}{b}+\frac{b}{x},$$

$$\left(\frac{1}{a}-\frac{1}{b}\right)x=\frac{b-a}{x},$$

$$\frac{b-a}{ab}x=\frac{b-a}{x}, \qquad \text{(A)}$$

and, b being supposed different from a,

$$\frac{x}{ab}=\frac{1}{x},$$

$$x^2=ab,$$

$$x=\pm\sqrt{ab}.$$

This is a pure quadratic (239).

The provision that b is different in value from a is required, because otherwise each side of (A) is zero, and the equation states no fact to characterize any special value of x. If $b=a$, the original equation becomes

$$\frac{x}{a}+\frac{a}{x}=\frac{x}{a}+\frac{a}{x},$$

and is not an equation defining one or more values of x, but an identity, as it is termed, holding true whatever the value of x be.

Ex. 19. When the equation is relieved of the fraction it becomes an adfected quadratic, such as (248) contemplates.

$$x-\frac{x^3-8}{x^2+5}=2,$$

$$x^3+5x-x^3+8=2x^2+10,$$

$$2x^2-5x=-2,$$

$$\text{whence} \quad x=2, \text{ or } \tfrac{1}{2}.$$

Ex. 22. When each member of the equation is squared.

$$(3x-2)(2x-3) = 144,$$
$$6x^2-13x = 138,$$

whence $x = 6$, or $-\frac{23}{6}$.

Ex. 23. $$\sqrt{2x+4} = 1+\sqrt{\frac{x}{2}+6}.$$

If each side be squared,

$$2x+4 = 1+2\sqrt{\frac{x}{2}+6}+\frac{x}{2}+6, \quad (84, 85)$$
$$2\sqrt{\frac{x}{2}+6} = \frac{3x}{2}-3.$$

When each side is squared,

$$4\left(\frac{x}{2}+6\right) = \frac{9x^2}{2}-9x+9,$$
$$8x+96 = 9x^2-36x+36,$$
$$9x^2-44x = 60,$$

whence $x = 6$, or $-\frac{10}{9}$.

Ex. 29. This equation is conveniently solved by treating $\frac{x+2}{x-1}$ as a single quantity, when it becomes substantially similar to Ex. 26.

$$\frac{x+2}{x-1}-\frac{x-1}{x+2} = \frac{7}{12},$$
$$\left(\frac{x+2}{x-1}\right)^2-1 = \frac{7}{12}\,\frac{x-1}{x+2},$$
$$\left(\frac{x+2}{x-1}\right)^2-\frac{7}{12}\,\frac{x-1}{x+2}+\left(\frac{7}{24}\right)^2 = 1+\left(\frac{7}{24}\right) = \left(\frac{25}{24}\right)^2,$$
$$\frac{x+2}{x-1} = \frac{7\pm25}{24} = \frac{4}{3}, \text{ or } -\frac{3}{4},$$

whence $x = 10$, or $\frac{5}{7}$.

Ex. 42. This is an equation for finding n, which occupies the place hitherto taken by x. When the given values of a and b are introduced we have

$$\left(1+\frac{n-1}{3}\right)\frac{n}{2} = 48,$$
$$(n+2)n = 228,$$
$$n^2+2n+1 = 289,$$
$$n+1 = \pm 17,$$
$$n = 16, \text{ or } -18,$$

and the positive integral value of n which is required is 16.

272. Ex. 1. Here x^2 is the quantity whose square and simple power appear.

$$5x^4 - 11x^2 = 306$$
$$(x^2)^2 - \tfrac{11}{5}x^2 + (\tfrac{11}{10})^2 = \tfrac{306}{5} + (\tfrac{11}{10})^2 = (\tfrac{79}{10})^2.$$
$$x^2 = \frac{11 \pm 79}{10} = 9 \text{ or } -\frac{34}{5},$$

and if real roots alone are to be given as solutions,

$$x = \pm 3.$$

Ex. 2. The quantity $x^{\frac{3}{2}}$, which signifies the cube of the square root of x, or, which is the same thing, the square root of the cube of x (32), is here the quantity whose square and simple power appear and constitute the quadratic equation, x^3 being $(x^{\frac{3}{2}})^2$.

$$x^3 - 7x^{\frac{3}{2}} + (\tfrac{7}{2})^2 = (\tfrac{7}{2})^2 + 8 = (\tfrac{9}{2})^2,$$
$$x^{\frac{3}{2}} = \frac{7 \pm 9}{2} = 8, \text{ or } -1,$$
$$x^3 = 64, \text{ or } 1,$$
$$x = 4, \text{ or } 1.$$

When 1 is presented as a root of this equation, it will not be found to satisfy the equation on substitution if we accept

1 as the square root of 1; but it is to be remembered that we have the option of taking -1 as the square root of 1 (93), and the equation is then satisfied.

Ex. 3. Let $\sqrt{x^2-9}$ be deemed the quantity in which the quadratic equation is formed with itself and its square (265).

$$x^2-9-\sqrt{x^2-9}=12$$
$$x^2-9-\sqrt{x^2-9}+\tfrac{1}{4}=\tfrac{49}{4},$$
$$\sqrt{x^2-9}=\frac{1\pm7}{2}=4, \text{ or } -3,$$
$$x^2=25, \text{ or } 18,$$
$$x=\pm5, \text{ or } \pm\sqrt{9\times2}=+3\sqrt{2}.$$

Ex. 4. $$\sqrt{x^2+1}+4=\frac{5}{\sqrt{x^2+1}}$$
$$x^2+1+4\sqrt{x^2+1}+4=9,$$
$$\sqrt{x^2+1}=-2\pm3=1 \text{ or } -5,$$
$$x^2=0, \text{ or } 24,$$
$$x=0, \text{ or } \pm\sqrt{4\times6}=+2\sqrt{6}.$$

Ex. 5. Since $x-1=(\sqrt{x}-1)(\sqrt{x}+1)$,
$$\therefore \sqrt{x}+1=x+\tfrac{5}{4},$$
$$x-\sqrt{x}+\tfrac{1}{4}=0,$$
$$\sqrt{x}-\tfrac{1}{2}=0,$$
$$\sqrt{x}=\tfrac{1}{2},$$
$$x=\tfrac{1}{4}.$$

Ex. 6. $$\sqrt{x^2-2x+9}-\frac{x^2}{2}=3-x,$$
$$x^2-2x+9-2\sqrt{x^2-2x+9}=3.$$

After the model of (269) we add 1 to each side of the

equation to complete the square, and

$$\sqrt{x^2-2x+9} = 1 \pm 2 = 3, \text{ or } -1.$$
$$x^2-2x+9 = 9, \text{ or } 1.$$
$$\text{I.}\quad x^2-2x = 0,$$
$$x = 0, \text{ or } 2.$$
$$\text{II.}\quad x^2-2x+8 = 0,$$

an equation which has no real roots (255).

287. Ex. 1. $(1)+(2)$ gives $(x+y)^2 = a^2+b^2$,

$$x+y = \pm\sqrt{a^2+b^2} \quad (3).$$

$$(1)\div(3) \text{ gives } x = \pm\frac{a^2}{\sqrt{a^2+b^2}}$$

$$(2)\div(3) \text{ gives } y = \pm\frac{b^2}{\sqrt{a^2+b^2}}$$

Ex. 2. Since $\dfrac{xy+1}{y} = \dfrac{xy+1}{x}$,

$$\therefore\ x = y,$$

and either equation then becomes

$$x+\frac{1}{x} = \frac{5}{2},$$

whence $x = 2$, or $\frac{1}{2}$. (261. Ex. 40.)

Ex. 3.
$$y = 4-x^2$$
$$\therefore\ (4-x^2)^2+x = 10,$$
$$x^4-8x^2+x+6 = 0,$$
$$x^4-2x^2+1 = 6x^3-x-5,$$
$$(x^2-1)^2 = (6x+5)(x-1), \quad \text{(A)}$$

which is satisfied by

$$x = 1,$$

and then $y = 3$.

This is a pair of roots which satisfies tne given equation, but there are others resulting from the rest of the roots of equation (A), if the methods were brought into use which

have been devised for the solution of equations of a higher degree, as they are termed, than quadratics, equations not restricted to contain the first and second forms of the quantity to be determined. This is not the complete solution of the pair of equations presented, but the solution so far as it is attained by the methods which the Text Book teaches.

Ex. 4. When the equations are placed in the forms

$$xy+1 = 5{\cdot}5y, \quad (1)$$
$$xy+1 = 2{\cdot}5x, \quad (2)$$

it appears that

$$5{\cdot}5y = 2{\cdot}5x,$$
$$\text{or } x = \tfrac{11}{5}y,\ xy = \tfrac{11}{5}y^2.$$

Then (1) gives

$$y^2 - \frac{5y}{2} + \frac{5}{11} = 0,$$

$$\text{whence } y = \frac{5\sqrt{11} \pm \sqrt{195}}{4\sqrt{11}},$$

$$\text{and then } x = \frac{55 \pm \sqrt{2145}}{20}.$$

Ex. 5. According to the method employed in (275) let $y = kx$,

$$\therefore\ x^2(1-k^2) = 7,$$
$$kx^2 = 12,$$
$$\therefore\ 1-k^2 = \tfrac{7}{12}k,$$
$$\text{and } k = \tfrac{3}{4}, \text{ or } -\tfrac{4}{3}.$$

1. If $k = \frac{3}{4}$, $x^2 = +16$, $x = \pm 4$, $y = \pm 3$,

$$\text{and } \left.\begin{aligned} x &= 4 \\ y &= 3 \end{aligned}\right\}, \qquad \left.\begin{aligned} x &= -4 \\ y &= -3 \end{aligned}\right\},$$

are a pair of solutions.

2. If $k = -\frac{4}{3}$, $x^2 = -9$,

and no numerical value of x or y can result.

The same method is applicable to Ex. 6.

Ex. 7. $$3x+2y=13, \quad (1)$$
$$xy=6. \quad (2)$$

$(1)^2-24\times(2)$ gives $3x-2y=\pm 5$. (3)

$(1)+(3)$ gives $x=3$, or $\frac{4}{3}$,

$(1)-(3)$ „ $y=2$, or $\frac{9}{2}$.

Ex. 8 and 9. These equations differ only from the earlier examples of (224) in having the squares of x and y instead of the simple powers.

$$4x^2+7y^2=148, \quad (1)$$
$$3x^2-y^2=11. \quad (2)$$

$(1)+7\times(2)$ gives $25x^2=225$, $x^2=9$, $x=\pm 3$.

Then $y^2=16$, $y=\pm 4$.

Since the double signs arise independently, we have the four pairs of solutions.

$$\left.\begin{array}{l} x=3 \\ y=4 \end{array}\right\},\ \left.\begin{array}{l} x=3 \\ y=-4 \end{array}\right\},\ \left.\begin{array}{l} x=-3 \\ y=4 \end{array}\right\},\ \left.\begin{array}{l} x=-3 \\ y=-4 \end{array}\right\}.$$

Ex. 10, 11, and 12 are similar in structure, the first giving a relation in the form of a simple equation between x and y, the second a simple equation between their reciprocals.

Ex. 10. The equations being

$$3x+2y=6, \quad (1)$$
$$3x+2y=4xy, \quad (2)$$
$$4xy=6,\ 2y=\frac{3}{x}.$$
$$\therefore x+\frac{3}{x}=6,$$
$$x^2-2x+1=0,$$
$$x=1,\ y=\frac{3}{2}.$$

Ex. 11 may be solved in the same manner.

Ex. 12. Consistently with (1) let

$$\left.\begin{aligned} x &= \frac{c}{2}+z \\ y &= \frac{c}{2}-z \end{aligned}\right\}.$$

$$\therefore \frac{a^2}{\frac{c}{2}+z}+\frac{b^2}{\frac{c}{2}-z}=\frac{(a+b)^2}{c},$$

$$\frac{(a+b)^2}{c}z^2+(a^2-b^2)z+\frac{(a-b)}{4}c=0,$$

$$\text{whence } z=-\frac{a-b}{a+b}\,\frac{c}{2}.$$

$$\therefore x=\frac{c}{2}+z=\frac{bc}{a+b},$$

$$y=\frac{c}{2}-z=\frac{bc}{a+b}.$$

Ex. 13.
$$x^4+y^4=337, \qquad (1)$$
$$xy=12. \qquad (2)$$

$(1)+2\times(2)^2$ gives $x^2+y^2=\pm 25$.
$(1)-2\times(2)^2$ gives $x^2-y^2=\pm 7$.

The double signs arising independently one of the other, we have

$$x^2=16 \text{ or } 9, \quad y^2=9 \text{ or } 16,$$
$$x=\pm 4 \text{ or } \pm 3, y=\pm 3 \text{ or } \pm 4;$$

and the sets of solutions

$$\left.\begin{aligned} x&=4\\ y&=3\end{aligned}\right\},\ \left.\begin{aligned} x&=-4\\ y&=3\end{aligned}\right\},\ \left.\begin{aligned} x&=4\\ y&=-3\end{aligned}\right\},\ \left.\begin{aligned} x&=-4\\ y&=-3\end{aligned}\right\},$$

$$\left.\begin{aligned} x&=3\\ y&=4\end{aligned}\right\},\ \left.\begin{aligned} x&=-3\\ y&=-4\end{aligned}\right\},\ \left.\begin{aligned} x&=3\\ y&=-4\end{aligned}\right\},\ \left.\begin{aligned} x&=-3\\ y&=-4\end{aligned}\right\}.$$

But of these eight pairs there are four, namely the 2nd, 3rd, 6th, and 7th, which are inconsistent with (2), and we have the remaining four pairs only admissible,

$$\left.\begin{array}{l} x=4 \\ y=3 \end{array}\right\}, \quad \left.\begin{array}{l} x=-4 \\ y=-3 \end{array}\right\}, \quad \left.\begin{array}{l} x=3 \\ y=4 \end{array}\right\}, \quad \left.\begin{array}{l} x=-3 \\ y=-4 \end{array}\right\}.$$

The reason of four pairs of solutions appearing, which we have to discard, is that in the process of solution we have used equation (2) in the form $x^2y^2 = 144$. Now the four discarded pairs of solutions satisfy this equation, but they do not satisfy $xy = 12$. Compare (283).

Ex. 14.
$$\text{Let } x-b = z,$$
$$\therefore y-a = z.$$

Then the first equation gives

$$z^2+(2a+b)z = 0,$$

$$\text{whence } z = 0,$$
$$\text{or } z = -2a-b.$$

$$\therefore \left.\begin{array}{l} x=b \\ y=a \end{array}\right\},$$
$$\text{or } \left.\begin{array}{l} x=-2a \\ y=-a-b \end{array}\right\}.$$

Ex. 15. This example may appear to present three equations, but they are in reality but two, because any two will produce the third.

$$\text{Since } x+y+3\sqrt{x+y} = 10,$$

if we treat $\sqrt{x+y}$ as a single quantity, of which the simple power and the square appear in this equation, the usual treatment of quadratics gives

$$x+y+3\sqrt{x+y}+\tfrac{9}{4}=\tfrac{49}{4},$$
$$\sqrt{x+y}=2, \text{ or } -\tfrac{5}{2},$$
$$x+y=4, \text{ or } \tfrac{25}{4}.$$

$$\text{Since } (x+y)^2+(x-y)^2=2(x^2+y^2)$$

the second equation gives

$$(x+y)^2+(x-y)^2=20,$$
$$\therefore\ (x-y)^2=4, \text{ or } \tfrac{195}{4},$$
$$x+y=\pm 2, \text{ or } \pm\tfrac{\sqrt{195}}{2}.$$

If the irrational roots be disregarded,

$$\left.\begin{array}{l} x=3 \\ y=1 \end{array}\right\}, \quad \text{or} \quad \left.\begin{array}{l} x=1 \\ y=3 \end{array}\right\}.$$

Ex. 16. When it is observed that

$$\frac{x^2}{y^2}+\frac{y^2}{x^2}=\left(\frac{x}{y}-\frac{y}{x}\right)^2+2,$$

the first equation takes the form

$$\left(\frac{x}{y}-\frac{y}{x}\right)^2+\frac{x}{y}-\frac{y}{x}=\frac{88}{9},$$

and if this be solved as a quadratic, wherein $\frac{x}{y}-\frac{y}{x}$ is treated as the quantity to be found,

$$\frac{x}{y}-\frac{y}{x}=\frac{-1\pm 19}{6}=3 \text{ or } -\frac{10}{3},$$
$$\therefore\ x^2-y^2=3xy, \text{ or } -\tfrac{10}{3}xy,$$
$$=9, \text{ or } -10.$$

The example may now be treated similarly to Ex. 5.

Ex. 17. If $x+y$ is regarded as a single quantity, its value can be found from the first equation. Then the example becomes but a simple form on the model on which Ex. 7 may be supposed to be constructed.

From the first equation

$$x+y = \tfrac{7}{14}, \text{ or } -8.$$

When $\left.\begin{aligned} x &= 4 \\ y &= 3 \end{aligned}\right\}$, $\left.\begin{aligned} x &= 3 \\ y &= 4 \end{aligned}\right\}$, $\left.\begin{aligned} x &= -2 \\ y &= -6 \end{aligned}\right\}$. $\left.\begin{aligned} x &= -6 \\ y &= -2 \end{aligned}\right\}$.

Ex. 18. $$\left.\begin{aligned} \sqrt{x^2-11}+\sqrt{y^2-5} &= 7 \\ x^2y^2-11y^2-5x^2 &= 45 \end{aligned}\right\}. \quad \begin{aligned} (1) \\ (2) \end{aligned}$$

The manner of solving this equation is suggested by observing that

$$(x^2-11)(y^2-5) = x^2y^2-11y^2-5x^2+55.$$

Hence (2) becomes

$$(x^2-11)(y^2-5) = 100,$$

$$\sqrt{x^2-11}\ \sqrt{y^2-5} = \pm 10. \quad (3)$$

It will now be easy to find the difference of the two quantities $\sqrt{x^2-11}$ and $\sqrt{y^2-5}$, of which we already know the sum in (1).

$(1)^2-4\times(3)$ gives

$$x^2-11-2\sqrt{x^2-11}\ \sqrt{y^2-5}+y^2-5 = 9, \text{ or } 89,$$

$$\sqrt{x^2-11}-\sqrt{y^2-5} = \pm 3, \text{ or } \pm\sqrt{89}. \quad (4)$$

$$(1)+(4) \text{ gives } 2\sqrt{x^2-11} = 10, \text{ or } 4, \text{ or } 7\pm\sqrt{89},$$

$$x^2-11 = 25, \text{ or } 4, \text{ or } \left(\frac{7\pm\sqrt{89}}{2}\right)^2,$$

$$x^2 = 36, \text{ or } 15, \text{ or } 11+\left(\frac{7\pm\sqrt{89}}{2}\right),$$

$$x = \pm 6, \text{ or } \pm\sqrt{15}, \text{ or } \pm\sqrt{11+\left(\frac{7\pm\sqrt{89}}{2}\right)^2}.$$

$$(1)-(4) \text{ gives } 2\sqrt{y^2-5} = 4, \text{ or } 10, \text{ or } 7\mp\sqrt{89},$$

$$y^2 = 9, \text{ or } 30, \text{ or } 5+\left(\frac{7\mp\sqrt{89}}{2}\right)^2,$$

$$y = \pm 3, \text{ or } \pm\sqrt{30}, \text{ or } \pm\sqrt{5+\left(\frac{7\mp\sqrt{89}}{2}\right)^2}.$$

Hence if the last values be not taken into account from the difficulty of expressing them in numbers, we have the eight pairs of solutions:

$$\left.\begin{array}{l} x = 6 \\ y = 3 \end{array}\right\}, \quad \left.\begin{array}{l} x = -6 \\ y = 3 \end{array}\right\}, \quad \left.\begin{array}{l} x = 6 \\ y = -3 \end{array}\right\}, \quad \left.\begin{array}{l} x = -6 \\ y = -3 \end{array}\right\},$$

$$\left.\begin{array}{l} x = \sqrt{15} \\ y = \overline{30} \end{array}\right\}, \quad \left.\begin{array}{l} x = -\sqrt{15} \\ y = \sqrt{30} \end{array}\right\}, \quad \left.\begin{array}{l} x = \sqrt{15} \\ y = -\sqrt{30} \end{array}\right\}, \quad \left.\begin{array}{l} x = -\sqrt{15} \\ y = -\sqrt{30} \end{array}\right\}.$$

Ex. 19. From the first equation

$$(x+\sqrt{x^2-y^2})^2+(x-\sqrt{x^2-y^2})^2$$
$$= 9(x+\sqrt{x^2-y^2})(x-\sqrt{x^2-y^2}),$$
$$4x^2-2y^2 = 9y^2,$$
$$\text{or } 4x^2 = 11y^2.$$

From the second

$$(\sqrt{x^2-y^2}+y)^2-(\sqrt{x^2-y^2}-y)^2$$
$$= 7(\sqrt{x^2-y^2}-y)(\sqrt{x^2-y^2}+y),$$
$$4y\sqrt{x^2-y^2} = 7(x^2-2y^2).$$

If the value $x^2 = \frac{11}{4}y^2$ obtained from the former equation is introduced,

$$4y\sqrt{\frac{7}{4}y^2} = 7\frac{3y^2}{4},$$

$$\text{or } 2\sqrt{7} = \tfrac{21}{4}, \text{ an untrue result.}$$

Hence the relation between x and y obtained from the former equation does not satisfy the second, or the equations are incompatible.

Ex. 20.
$$\left.\begin{aligned} xy+xz &= 5 \\ xy+yz &= 1 \\ xz+yz &= 9 \end{aligned}\right\}.$$

Regard xy, xz, yz as three separate quantities, and the equations have the form of Ex. 1 in 229, and will give

$$yz = 6,\ xz = 3,\ xy = 2,$$
$$\text{whence } x^2y^2z^2 = 36,$$
$$xyz = \pm 6.$$

$$\therefore x = \frac{xyz}{yz} = \pm 1,$$
$$y = \pm 2,$$
$$z = \pm 3,$$

and the conditions of the products being positive require that all the positive roots shall be taken together, and all the negative roots together.

Ex. 21. The following method of solving this example is suggested by observing how $x+z$ appears as a single quantity in the second and third equations.

In compliance with the last equation let

$$\left.\begin{aligned} x+z &= u+1 \\ y &= u-1 \end{aligned}\right\}.$$

From the second equation

$$\begin{aligned} xz &= 2y(x+z)-13, \\ &= 2(u^2-1)-13, \\ &= 2u^2-15. \end{aligned}$$

From the first equation

$$(x+z)^2-2xz-y^2 = 6,$$
$$(u+1)^2-2(2u^2-15)-(u-1)^2 = 6.$$
$$u^2-u-6 = 0,$$
$$u = 3, \text{ or } -2.$$

1. If $u = 3$, $y = 2$,

$$x + z = 4,$$
and
$$xz = 3,$$
whence
$$x = 1, \text{ or } 3,$$
$$z = 3, \text{ or } 1.$$

2. If $u = -2$, $y = -3$,

$$x + z = -1,$$
and
$$xz = -7,$$
whence
$$x = \frac{-1 \pm \sqrt{29}}{2},$$
$$z = \frac{-1 \pm \sqrt{29}}{2}.$$

301. Ex. 2. Let x be the middle number, and therefore $x-1$ the preceding, $x+1$ the succeeding number. The question then states that

$$(x-1)x(x+1) = 3x.$$

This equation is satisfied by $x = 0$, when the numbers would appear to be -1, 0, 1; but this solution is irrelevant because 0 cannot be taken as a number in the sense intended. Hence to determine x we have

$$(x-1)(x+1) = 3,$$
$$x = \pm 2.$$

The numbers then are 1, 2, and 3,
or -3, -2, and -1;

and the latter group may be disregarded if, as may be assumed, positive numbers are expected as the solution of the question.

Ex. 3. The number may be expressed by $2x \times 10 + x$ or $21x$, and when the digits are inverted it is $x \times 10 + 2x$ or $12x$.

Hence
$$21 + 12x^2 = 2268,$$
$$x = \pm 3.$$

The number therefore is 63 or -63, and the latter solution may be declined on supposition of a positive number being expected.

We have instances in these problems to exemplify the remark in (290), that the algebraical statement of a problem may be more comprehensive than the verbal statement.

Ex. 4. Suppose that the farmer buys $x-3$ sheep. He then pays $\frac{72}{x-3}$£ for each. Had he bought $x+3$, he would have paid $\frac{72}{x+3}$£ for each.

Hence by the question,

$$\frac{72}{x-3}-\frac{72}{x+3}=1,$$

$$\text{whence } x=\pm 21.$$

He therefore buys 18 or -24 sheep.

The number of sheep being necessarily positive, the latter solution is dismissed, being an algebraical result which the equation embraces along with the result which we seek to answer the question proposed.

Ex. 5. Let one train run $x+5$ and the other $x-5$ miles an hour.

They run 1200 miles in $\frac{1200}{x+5}$ and $\frac{1200}{x-5}$ hours respectively.

$$\therefore \frac{1200}{x-5}-\frac{1200}{x+5}=10,$$

$$x=\pm 35,$$

and the rates are 30 and 40 miles an hour, the negative solution being irrelevant.

Ex. 6. Let a be the cost price.

The selling price is $\left(1+\frac{n+50}{100}\right)a$,

and the profit under $\frac{n+50}{100}a$.

Now this profit is n per cent on the selling price.

$$\therefore \frac{n+50}{100}a = \frac{n}{100}\left(1+\frac{n+50}{100}\right)a,$$

$$\text{or } n+50 = n.\frac{n+150}{100},$$

$$n = 50 \text{ or } -100.$$

The former value alone is applicable to the question.

Ex. 7. The sides being $x-1$ and $x+1$ yards in length,

$$\begin{aligned}(10)^2 &= (x-1)^2+(x+1)^2,\\ &= 2x^2+2,\\ x &= \pm 7,\end{aligned}$$

and the sides are 6 and 8 yards in length.

Ex. 8. Let one piece be x yards long, its price x shillings the yard, and its value therefore x^2 shillings.

Let the other be $x+3$ yards long, its price $x+3$ shillings the yard, and its value therefore $(x+3)^2$ shillings.

The condition of the question gives

$$x^2+(x+3)^2 = 89,$$

$$\text{whence } x = 5 \text{ or } -8.$$

The negative root being disregarded, the lengths of the pieces are 5 and 8 yards respectively.

Ex. 9. Let the runner who starts from A run x yards a minute,

and „ „ „ B run y yards a minute.

Since when they are running to meet one another they lessen the distance between themselves by $x+y$ yards every minute, and as they meet in 6 minutes, their original separation, the length AB, must be $6(x+y)$ yards.

The runner who starts from A performs this distance in $\frac{6(x+y)}{x}$ and the other in $\frac{6(x+y)}{y}$ minutes.

$$\therefore \frac{6(x+y)}{x} - \frac{6(x+y)}{y} = 3\tfrac{1}{2},$$

$$\text{or } \frac{y}{x} - \frac{x}{y} = \frac{7}{12}.$$

All the given conditions being now embodied in Algebra, we have not the means of determining x and y, since one equation only arises. But the question demands not the actual rates of running but a comparison of these rates, or the fraction $\frac{y}{x}$.

$$\text{Now } \left(\frac{x}{y}\right)^2 - \frac{7}{12}\frac{y}{x} = 1,$$

$$\therefore \frac{y}{x} = \frac{4}{3}, \text{ or } -\frac{3}{4}.$$

Rejecting the latter solution, we have the result that $y = \frac{4}{3}x$ or $x = \frac{3}{4}y$, so that the pace of the slower man is $\frac{3}{4}$ of the pace of the faster.

LOGARITHMS.

It is hoped that the examples offered for practice in the Chapter on Logarithms follow so directly those which are fully worked in the text, that no further assistance will be required.

The late Professor De Morgan recommended for many purposes the use of logarithms computed to four places of decimals only instead of to seven. Where greater exactness is not required, these four-figure logarithms have the advantage of being used more expeditiously, because the logarithms of all numbers can then be taken from tables printed on a card about twice the size of this page. It may be interesting to see the degree of accuracy to which four-figure logarithms give a result in the instance of a ball (46).

With four-figure logarithms the work will stand

$$\begin{aligned} \log 14{\cdot}26 &= 1{\cdot}1541 \\ &\quad\;\; \underline{3} \\ \log (14{\cdot}26)^3 &= 3{\cdot}4623 \\ \log {\cdot}016887 &= \underline{\bar{2}{\cdot}2276} \\ \text{sum} &= 1{\cdot}6899 \\ \log 48{\cdot}97 &= 1{\cdot}68 \end{aligned}$$

so that the inaccuracy is not $\frac{1}{100}$ of a cubic inch.

As another application of these four-figure logarithms Ex. 1 of (60) shall be solved by them.

The expression to be computed is

$$\frac{100}{\cdot 04}\left\{1-\left(\frac{1}{1\cdot 04}\right)^{10}\right\}.$$

$$\begin{aligned}
\text{Now } \log 1\cdot 04 &= \cdot 017,\\
\log \frac{1}{1\cdot 04} &= -\cdot 017,\\
\log \left(\frac{1}{1\cdot 04}\right)^{10} &= -\cdot 17 = \bar{1}\cdot 83,\\
\log \cdot 6761 &= \bar{1}\cdot 83,\\
\therefore 1-\left(\frac{1}{1\cdot 04}\right)^{10} &= \cdot 3239,
\end{aligned}$$

and the value of the lease is $\frac{32\cdot 39}{\cdot 04} = £809\ 15s.\ 0d.$

The error through the use of these small tables amounts to £1 5*s.* 0*d.*

TRIGONOMETRY.

CHAPTER II.

51. Ex. 3, 4, 5. Since $\frac{1}{4} = \cdot 25$, $\frac{1}{3} = \cdot 3333333$, $\frac{1}{6} = \cdot 1666667$, when the fractions are thus converted into decimals the tables can be employed to give the angles required.

Ex. 7. If A be the angle,

$$\sin A = \sqrt{\tfrac{7}{15}}.$$

$$\begin{aligned}
\log 7 &= \phantom{\bar{1}}\cdot 845\,0980 \\
\log 15 &= 1\cdot 176\,0913 \\
\log \tfrac{7}{15} &= \bar{1}\cdot 669\,0067 \\
\log \sqrt{\tfrac{7}{15}} &= \bar{1}\cdot 834\,5034 \\
\text{L} \sin A &= 9\cdot 834\,5043,
\end{aligned}$$

$$\text{whence } A = 43^\circ\ 5'\ 19'',$$

of which the supplement, the other angle required, is $136^\circ\ 54'\ 41''$.

Ex. 9. The tables give $\text{L} \sin 72^\circ\ 16' = 9\cdot 978\,8579$

$$\begin{aligned}
&\text{prop. part. for } 52'' && \quad\quad\ \ 354 \\
\therefore\ &\text{L} \sin 72^\circ\ 16'\ 52'' &&= 9\cdot 978\,8933 \\
\therefore\ &\log \sin 72^\circ\ 16'\ 52'' &&= \bar{1}\cdot 978\,8933 \quad (43) \\
\therefore\ &\log \sqrt[3]{\sin 72^\circ\ 16'\ 52''} &&= \bar{1}\cdot 992\,9644 \\
&\log \cdot 98393 &&= \bar{1}\cdot 992\,9642, \\
\therefore\ &\sqrt[3]{\sin 72^\circ\ 16'\ 52''} &&= \cdot 98393.
\end{aligned}$$

Ex. 10. From the tables $1 + \sin 10° \; 24 = 1{\cdot}1805191$.

$$\text{If } x = \frac{1}{1{\cdot}1805191 . \sin^2 79° \; 36'},$$

$$-\log x = \log 1{\cdot}1805191 + 2\text{L} \sin 79° \; 30 - 20.$$

$$\begin{array}{rr}
= \; 9{\cdot}992 & 8059 \\
 & 2 \\ \hline
19{\cdot}985 & 6118 \\
{\cdot}072 & 0660 \\
 & 37 \\
 & 33 \\ \hline
{\cdot}057 & 6848 \\
\log x = \bar{1}{\cdot}942 & 3152
\end{array}$$

$x = {\cdot}8756$ to the fourth place of decimals.

Ex. 11. Since $\text{L} \tan 15° = 9{\cdot}4280525$,

$$\log \tan 15° = \bar{1}{\cdot}4280525 \quad (43)$$

$$\begin{array}{lrr}
\log 16{\cdot}666 = & 1{\cdot}221 & 8314 \\
\text{prop. part for } 7 & & 183 \\ \hline
 & 1{\cdot}221 & 8497 \\
 & \bar{1}{\cdot}428 & 0525 \\ \hline
 & 2)1{\cdot}793 & 7972 \\ \hline
 & {\cdot}896 & 8986 = \log 7{\cdot}8868.
\end{array}$$

Ex. 12. Since $\tan 129° \; 18' = -\tan 50° \; 42'$ (25)

$$\begin{array}{rrl}
\text{L} \tan 50° \; 42' = & 10{\cdot}086 & 9863 \\
\log \tan 50° \; 42' = & {\cdot}086 & 9863 \\
\log (\tan 50° \; 42')^{\frac{1}{5}} = & {\cdot}017 & 3973 = \log 1{\cdot}0409,
\end{array}$$

$$\therefore \; (\tan 129° \; 18')^{\frac{1}{5}} = -1{\cdot}0409.$$

CHAPTER III.

66. Ex. 3. Let AB be the hypothenuse of the triangle ABC. Bisect it in O, so that $AO=BO=\frac{73}{2}$ yards. With centre O describe a semicircle, and C will fall on the circumference (*Euclid* iii. 31, or *Text Book on Geometry* ii. 13). From C draw CD perpendicular to AB, which, if A is the smaller of the two acute angles, lies between O and B. Then COB, as the angle at the centre, is twice CAB the angle at the circumference on the same base (*Euclid* iii. 20, or *Text Book on Geometry* ii. 10), and $\sin COB = \frac{CD}{CO}$.

Now $73 \times CD$ is twice the area of the triangle,

$$\therefore CD = \frac{1936}{73} \text{ feet.}$$

$$\therefore \sin COB = \frac{2 \times 1936}{(73)^2} = \frac{3872}{5329}.$$

$$\begin{aligned} L \sin COB &= 13{\cdot}587\,9353 \\ &\underline{3{\cdot}726\,6457} \\ &9{\cdot}861\,2896. \end{aligned}$$

$$\therefore COB = 46^\circ\ 36',$$

whence $CAB = 23^\circ\ 18'$, and $CBA = 66^\circ\ 42'$.

CHAPTER IV.

95. The sides of a triangle follow as to order of magnitude the angles which are opposite to them (*Euclid* i. 19, or *Text Book on Geometry* i. 8). Hence in some of these examples, when the longest side is required, it is the side

opposite to the largest of the three angles; the side neither the longest nor shortest will be that opposite to the angle which is neither the largest nor smallest of the three. Conversely the angles follow as to order of magnitude the sides opposite to them.

100. Ex. 2. When a triangle is equilateral,

$$S \text{ becomes } \frac{3a}{2},$$

$$S-a=S-b=S-c \text{ each becomes } \frac{a}{2},$$

$$\text{and the area is } \tfrac{1}{4}\sqrt{3}\,a^2.$$

This expression arises also from observing that as each angle of the triangle is 60°, the perpendicular from an angle on any side a is $a \sin 60° = \frac{\sqrt{3}}{2}a$, and the area being half the product of the perpendicular and that side is $\frac{\sqrt{3}}{4}a^2$

In this instance $a = 17{\cdot}04$ chains,

$$\text{and the area} = (8{\cdot}52)^2\sqrt{3} = 125{\cdot}7 \text{ square chains,}$$
$$= 12{\cdot}57 \text{ acres.}$$

Ex. 6. Let $AOB = 637$ feet be one diagonal, and $COD = 598$ feet be the other, O being their intersection, and the angle COB or AOD being 37° 18′. From C draw CE, and from D draw DF, each perpendicular to AB. Then the quadrilateral consists of two triangles ACB, ADB.

The area of $ACB = \frac{1}{2} AB \,.\, CE$
$= \frac{1}{2} AB \,.\, CO \sin 37° \; 18'.$

" $ADB = \frac{1}{2} AB \,.\, DO \sin 37° \; 18'.$

$\therefore$ " the quadrilateral $= \frac{1}{2} AB\,(CO+DO) \sin 37° \; 18',$
$= \frac{1}{2} AB \,.\, CD \sin 37° \; 18',$
$= \frac{1}{2}\, 637 \times 598 \sin 37° \; 18',$
$= 115418{\cdot}6$ square feet.

102. In reference to the figure of (101), the area to be found is the triangle CAA', by which CBA' exceeds CBA.

Now $CB = 12$ feet, $CA = 8$ feet, angle $CBA = 30°$.

$$\therefore CD = 12 \sin 30° = 6 \text{ feet.}$$

$$\therefore AD = \sqrt{CA^2 - CD^2}$$

$$= \sqrt{64 - 36} = \sqrt{28} = 5{\cdot}2915 \text{ feet.}$$

$$\therefore \text{triangle } CAA' = AD \, . \, DC = 31{\cdot}749 \text{ square feet.}$$

CHAPTER V.

114. Ex. 1. Since CAB is a right angle, AB will be known if CA and CB are known, and these lengths can be found from the given lengths of the poles and the angular elevations of their summits.

$$CB = \frac{30}{\sin 28° \; 14'} = 63{\cdot}416 \text{ feet,}$$

$$CA = \frac{30}{\sin 34° \; 18'} = 53{\cdot}236 \text{ feet,}$$

$$\text{whence } AB = \sqrt{CB^2 - CA^2} = 34{\cdot}5 \text{ feet.}$$

Ex. 2. If x be the height in feet,

$$x = 325 \tan 8° \; 16' = 47{\cdot}22.$$

		Yards.
Ex. 3. Height of highest point	$= 400 \tan 49° \; 27'$ =	$467{\cdot}5$
" lowest "	$= 400 \tan 38° \; 51'$ =	$322{\cdot}1$
	Length of the line =	$145{\cdot}4$

145. Ex. 1. The conditions of the problem make ABC to be a triangle wherein C is the right angle, and AB is 250 yards in length, the angle CAB is $34° \; 18'$, and the angle ABC consequently is $55° \; 42'$.

The required distance $AC = AB \sin 55° \, 42'$,
$= 250 \sin 55° \, 42'$,
$= 206{\cdot}5$ yards.

Ex. 3. From the known length AC and the angle BAC, the length of BC can be found, and then from the angle BPC, CP can be found. Thence AP, the excess of AC over PC, is known.

$$BC = 600 \tan 20° \, 34',$$
$$PC = BC \tan 35° \, 41',$$
$$\therefore PC = 600 \tan 20° \, 34' \tan 35° \, 41'.$$

$$\begin{aligned} \log PC &= 2{\cdot}778\,1513 \\ &\quad\; 9{\cdot}574\,2761 \\ &\quad\; \underline{9{\cdot}856\,2042} - 20 \\ &\quad\; 2{\cdot}208\,6316 \end{aligned}$$

$$PC = 161 \text{ yards},$$
$$AP = 600 - 161 = 439 \text{ yards}.$$

Ex. 4. If x is the required height in yards of the base of the tower above the observer's eye, and d his distance horizontally from the tower,

$$30 + x = \text{height of summit} = d \tan 27° \, 14,$$
$$x = \quad \text{"} \quad \text{base} = d \tan 20° \, 13'.$$

$$\therefore \frac{30+x}{x} = \frac{\tan 27° \, 14'}{\tan 20° \, 13'} = 1{\cdot}397.$$

$$\therefore x = \frac{30}{{\cdot}397} = 75{\cdot}5 \text{ yards}.$$

Ex. 6. Since the length AB is known, the height of A above B will depend upon the inclination of AB to the horizon. This, therefore, is the element which we aim at determining.

Produce AB to meet in D a horizontal line through C, and from B draw BE perpendicular to this same horizontal

line. BE is then 50 feet, and BDC is the inclination of AB to the horizon.

Since AB and two angles of the triangle ABC are given, this triangle can be solved. The angle ACB is a right angle, since CAB and CBA together make a right angle.

$$BC = AB \sin 30° = 590 \text{ yards} = 1770 \text{ feet.}$$

The position of the triangle is now to be ascertained from the fact that BE is 50 feet.

$$\text{Sin } BCD = \frac{BE}{BC} = \frac{50}{1770}.$$

$$\therefore BCD = 1° \; 37' \; 5''.$$

$$\therefore BDC = 60° - BCD = 58° \; 22' \; 45''.$$

$$\therefore \text{the height of } A \text{ above } B = AB \sin BDC$$
$$= 3540 \sin 58° \; 22' \; 45''$$
$$= 3014 \text{ feet.}$$

Ex. 7. If CD is drawn perpendicular to AB, we have the means of determining the portions AD, BD, and thence the whole AB.

Since the angle $ACD = 55° \; 43'$ and the angle $BCD = 43° \; 31'$,

$$AD = 90 \tan 55° \; 43' = 103{\cdot}012 \text{ yards,}$$
$$BD = 90 \tan 43° \; 31' = \underline{85{\cdot}455} \quad ,,$$
$$\therefore AB = 217{\cdot}5 \text{ yards.}$$

Ex. 8. 1. In the triangle DBA we have the given parts

$$AB = 176 \text{ yards,}$$
$$\text{angle } CAB = 34° \; 18',$$
$$\text{angle } ABD = 103° \; 42',$$

from which AD can be found $= 255{\cdot}54$ yards.

2. In the triangle ACB we have the given parts

$$AB = 176 \text{ yards,}$$
$$\text{angle } CAB = 34° \; 18',$$
$$\text{angle } CBA = 76° \; 18',$$

from which AC can be found $= 182{\cdot}67$ yards.

Then $CD = AD - AC$ is known $= 72{\cdot}87$ yards.

Ex. 9. For the purpose of finding the length AB, the given length PQ is a superfluous condition.

From C draw CD perpendicular to AB.

Since PCB, QCA are equally inclined to PQ and therefore to CD, CAB is an isosceles triangle, and

$$\begin{aligned} AD = DB &= DC \tan 36^\circ\ 42' \\ &= 100 \tan 36^\circ\ 42' \\ &= 74{\cdot}54 \text{ feet.} \\ \therefore AB &= 149{\cdot}08 \text{ feet.} \end{aligned}$$

The knowledge of the length PQ would enable us, if it were desired, to find the distance of the road PQ from C.

Ex. 11. Since A has the same elevation at C and D, C and D have the same distance from the base B, or CBD is an isosceles triangle where the base is 328 feet and the angle at the vertex is 110°.

$$\begin{aligned} \text{Hence } BC &= \frac{164}{\sin 55^\circ}. \\ \text{Then } AB &= BC \tan 22^\circ\ 18' \\ &= 164 \frac{\tan 22^\circ\ 18'}{\sin 55^\circ}. \\ \log AB &= 2{\cdot}214\,8438 \\ &\underline{9{\cdot}612\,9214} \\ &11{\cdot}827\,7652 \\ &\underline{9{\cdot}913\,3645} \\ &1{\cdot}914\,4007 \\ AB &= 82{\cdot}1 \text{ feet.} \end{aligned}$$

Ex. 12 and 13. It may be useful to present some details respecting this solid, the tetrahedron or triangular pyramid, at greater length than the mere solution of these two examples may require. The reader who finds difficulty in seeing the relations of the parts of such a figure may be greatly assisted by forming a model in wood or cardboard.

In a regular triangular pyramid, every face (144) is the

same equilateral triangle, and every edge has the same length. Any one of the faces may be taken as base, or supposed to be that face on which, to assist our conception, the solid stands on a table. Let ABC be this face, and D the remaining angular point of the pyramid, or its vertex when it is imagined to stand upon ABC.

We have therefore the four triangular faces

$$ABC, \ DAB, \ DBC, \ DAC,$$

each an equilateral triangle of the same size, and the six edges,

$$DA, \ DB, \ DC, \ AB, \ BC, \ CA,$$

which are straight lines of the same length.

We can determine by calculation

(1) The angle included between two faces.

(2) The angle included between two edges at the point where they meet.

(3) The angle between an edge and the face which it meets.

From the point D draw a straight line perpendicular to the plane ABC, and let it meet ABC in O. This line DO shall be used as an axis or line of reference, and other lines shall be determined in position by their inclinations to it.

Join AO, BO, CO. Since DO is perpendicular to the plane ABC, it is perpendicular to each of the straight lines AO, BO, CO. (*Euclid* xi. *Def.*)

Since then $DA^2 = DB^2 = DC^2$,

$$AO^2 + OD^2 = BO^2 + OD^2 = CO^2 + OD^2.$$

$$\therefore \ AO = BO = CO.$$

$$\text{Hence angle } OBC = \text{angle } OCB,$$

$$\text{angle } OBA = \text{angle } OCA,$$

$$\text{and angle } OAB = OAC.$$

(1) If then AO be produced to meet BC in E, AOE bisects BC at right angles. So also does DE.

Since then $\left.\begin{array}{l} AE \text{ in the plane } ABC \\ DE \quad ,, \qquad ,, \quad DBC \end{array}\right\}$ are perpendicular to BC,

AED is the inclination of these planes.

Now the angles of the equilateral triangle being each 60°, and being bisected by the lines OA, OB, OC,

$$OE = OB \sin 30° = \tfrac{1}{2}\, OB = \tfrac{1}{2}\, AO = \tfrac{1}{3}\, AE.$$

$$\therefore \sin EDO = \frac{OE}{DE} = \frac{OE}{AE} = \frac{1}{3},$$

$$EDO = 19° \; 28' \; 16'', \; (51)$$

$$\therefore AED = 70° \; 31' \; 44'.$$

This then is the magnitude of the angle between two faces of the tetrahedron.

(2) The edges which meet one another include the angle 60°.

(3) The angle between any edge and the face which it meets is DAO.

$$\text{Sin } ADO = \frac{AO}{AD} = \frac{AO}{AB}.$$

$$\text{Now } AO = \frac{2}{3}\, AE = \frac{2}{3}\, AB \sin 60° = \frac{\sqrt{3}}{3}\, AB$$

$$\therefore \sin ADO = \frac{\sqrt{3}}{3} = \cdot 5773503,$$

$$\therefore ADO = 35° \; 15' \; 52'',$$

$$\text{and } DAO = 54° \; 44' \; 8''.$$

If the figure described in Ex. 13, have its vertex denoted by D and its base by ABC, and if DO be drawn perpendicular to the base, it will be equally true as in the previous example that AO, BO, CO are equal in length, that they bisect the several angles of the equilateral triangle ABC, and that DE and AE each bisect at right angles the edge BC. Also in the right angled triangle BDC, $DE = BE = \tfrac{1}{2}\, AB$.

(1) $\text{Sin } ODE = \frac{OE}{DE} = \frac{1}{3}\frac{AE}{DE} = \frac{1}{3} \cdot \frac{AB \sin 60^\circ}{\frac{1}{2} AB} = \frac{\sqrt{3}}{3}.$

$$\therefore ODE = 35^\circ\ 15'\ 52'',$$
$$\therefore DEO = 54^\circ\ 44'\ \ 8',$$

and this is the inclination to the base of one of the three faces which meet in the vertex.

The faces which meet in the vertex are perpendicular to one another.

(2) Every two edges which meet in the vertex include a right angle, and each edge which meets the base makes with a side of the base the angle 45°.

(3) By the description of the figure

$$AO = \frac{2}{3} AE = \frac{2}{3} AB \sin 60^\circ = \frac{\sqrt{3}}{3} AB,$$

$$AD = \frac{AB}{\sqrt{2}}.$$

$$\therefore \sin ADO = \frac{AO}{AD} = \frac{\sqrt{6}}{3},$$

whence the angle ADO is $54^\circ\ 44'\ 15''$, and the angle DAO, which is the inclination of any edge DA to the base is $35^\circ\ 15'\ 45''$.

CHAPTER VI.

154. In Ex. 4 the sector will be a sixth part, and in Ex. 5 an eighth part of the respective circles.

Ex. 6. The area of the sector is 6 square feet (151).

The area of the circle is 36π square feet.

The angle of the sector must be the same fraction of 360° that 6 is of 36π,

or the angle of the sector is $\frac{60}{\pi}$ degrees$=19^\circ\ 5'\ 53''$.

Ex. 7. The area of the quadrant is $\frac{\pi}{4}$ or ·785 square yards.

The area of the right angled triangle formed by the chord and the two radii is $\frac{1}{2}$ or ·5 square yards.

Hence the area required between the chord and the arc is ·285 square yards.

164. Ex. 1. If we represent the sides by a, b, c, and suppose

$$\begin{aligned} a &= 329 \\ b &= 340 \\ c &= \underline{331} \\ 2S &= 1000, \end{aligned}$$

$$\begin{array}{rlrlrl} S = & 500 & S = & 500 & S = & 500 \\ a = & \underline{329} & b = & \underline{340} & c = & \underline{331} \\ S - a = & 171 & S - b = & 160 & S - c = & 169. \end{array}$$

The example shall now be worked as an instance of the use of four-figure logarithms.

$$\begin{aligned} \log(S - a) &= 2{\cdot}2330 \\ \log(S - b) &= 2{\cdot}2041 \\ \log(S - c) &= \underline{2{\cdot}2279} \\ & 6{\cdot}6650 \\ \log S &= \underline{2{\cdot}6990} \\ & 3{\cdot}9660 \end{aligned}$$

$$\log\sqrt{\frac{(S-a)(S-b)(S-c)}{S}} = 1{\cdot}9830 = \log 95{\cdot}57.$$

Thus in this example the use of the curtailed logarithms has given a result for the length of the radius which is a little more than 7 inches short of the true value.

Ex. 2. The diameter is $\dfrac{100}{\sin 27^\circ\ 34'}$

$$= \frac{100}{{\cdot}4627804} = 216 \text{ feet}.$$

Ex. 3. The hypotenuse of the triangle is $\sqrt{(17)^2 + (12)^2}$ or $20{\cdot}80865$ feet.

Now this hypotenuse is the diameter of the circumscribing circle (*Euclid* iv. 5. *Cor.*)

The radius therefore of the circumscribing circle is $10{\cdot}40433$ feet and its area $(10{\cdot}40433)^2\pi$ or 340 square feet.

Ex. 4. If R be the radius of the circle, a side of the triangle is $R\sqrt{3}$ (162), and the area of the triangle is $\frac{1}{2}(R\sqrt{3})^2 \sin 60^\circ$ (96). This area being a square yard,

$$1 = \frac{\sqrt{3}}{4}(R\sqrt{3})^2 = \frac{\sqrt{27}}{4}R^2,$$

$$\text{whence } R = \frac{2}{\sqrt[4]{27}} = {\cdot}8774 \text{ yards, or } 2{\cdot}63 \text{ feet.}$$

Ex. 5. If a be either of the shorter sides, the hypotenuse is $a\sqrt{2}$ feet, and the perimeter is $2a + a\sqrt{2}$ feet.

The area of the triangle being $\frac{1}{2}a^2$ bv the method of (156) we have, since r is 1 foot,

$$\frac{1}{2}a^2 = \frac{1}{2}(2a + a\sqrt{2})$$

$$\text{or } a = 2 + \sqrt{2} = 3{\cdot}414 \text{ feet.}$$

Ex. 6. The hypotenuse is $\sqrt{(21)^2 + (28)^2} = 35$ feet, and the radius of the inscribed circle $\dfrac{21 \times 28}{21 + 28 + 35}(157) = 7$ feet, whence the circumference is $43{\cdot}98$ or 44 feet nearly.

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